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An upper bound for the permanent of (0, 1)-matrices☆

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Abstract

A novel upper bound for the permanent of (0, 1)-matrices is obtained in this paper, by using an unbiased estimator of permanent [Random Structures Algorithms 5 (1994) 349]. It is a refinement of Minc's very famous result, and apparently tighter than the current best general bound in some cases.

Keywords: Permanent; (0, 1)-matrix; Upper bound; Unbiased estimator

1. Introduction

Let $A = [a_{ij}]$ be an $n \times n$ matrix with 0-1 entries, which is called a (0, 1)-matrix for briefness. Its permanent is defined as

$$\mathrm{Per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the sum goes over every permutation σ of the set $\{1, 2, \ldots, n\}$. Per(A) looks similar to the determinant of matrices. However, it is much harder to be computed. Valiant [6] proves that evaluating the permanent of a (0, 1)-matrix is a #P-complete

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problem. The following are the two most well known upper bounds for the permanent of (0, 1)-matrices.

Theorem 1.1 [3]. *Let $A = [a_{ij}]$ be a (0, 1)-matrix of order n. Its row sums are defined by $r_i = \sum_{j=1}^{n} a_{ij},\ i = 1, 2, \ldots, n$. Then*

$$\mathrm{Per}(A) \leqslant \prod_{i=1}^{n} \frac{r_i + 1}{2}.$$

Theorem 1.2 [1]. *Let $A = [a_{ij}]$ be a (0, 1)-matrix with row sums $r_1, r_2, \ldots, r_n$. Then*

$$\mathrm{Per}(A) \leqslant \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Theorem 1.2 is the best upper bound known for the permanent of (0, 1)-matrices. It was conjectured by Minc in 1963 [3]. The bound given by Theorem 1.2 is tighter than that of Theorem 1.1. A novel upper bound is obtained in this paper. Our tool is an unbiased estimator for the permanent of (0, 1)-matrices given by Rasmussen [5]. The new upper bound is a refinement of the result of Theorem 1.1, the very famous Minc bound, and sharper than that of Theorem 1.2 in some special cases.

2. Rasmussen's estimator (RAS)

Let $A(i, j)$ be the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column from the matrix A; and $A(i, :)$ be the ith row of matrix A. For any set S, let $|S|$ be the number of its elements. Algorithm 2.1 gives Rasmussen's unbiased estimator for permanent [5].

Algorithm 2.1 (RAS)

Input: A—an $n \times n$ (0, 1)-matrix.
Output: X_A—the estimation of Per(A).
step0: Let $p_i = 0$ for $i = 1, \ldots, n$;
step1: For $i = 1$ to n

If $|A(1, :)| = 0$, goto step2;

Choose a_{1j} from the nonzero elements of $A(1, :)$ uniformly at random;

Let $p_i =| A(1, :) |$;

$A = A(1, j)$;

End;

step2: $X_A = p_1 \times \cdots \times p_n$.

Through one stochastic experiment of Algorithm 2.1, one obtains either a permutation σ of $\{1, 2, \ldots, n\}$ such that $a_{i,\sigma(i)} = 1$ for all $i = 1, 2, \ldots, n$, or a permutation σ' of a subset of $\{1, 2, \ldots, n\}$ such that $(\sigma'(1), \ldots, \sigma'(j))$, $j < n$. We call the permutation obtained in this way a "random path". X_A given by Algorithm 2.1 is called random path value. It defines a random variable. A random path σ is said to be feasible if $X_A(\sigma) \neq 0$. Note that permutations σ which satisfy $\prod_{i=1}^{n} a_{i,\sigma(i)} = 1$ are one-to-one correspondent to feasible paths.

Theorem 2.1. *Let X_A be the random variable given by Algorithm* 2.1. *Then*

$$E[X_A] = \mathrm{Per}(A).$$

Proof. For a feasible path $\sigma = (j_1, j_2, \ldots, j_n)$, one can get

$$\begin{aligned} P[\sigma = (j_1, j_2, \ldots, j_n)] &= P[\sigma(1) = j_1, \ldots, \sigma(n) = j_n] \\ &= P[\sigma(1) = j_1] \cdot P[\sigma(2) = j_2 \mid \sigma(1) = j_1] \cdots \\ &\quad\; P[\sigma(n) = j_n \mid \sigma(1) = j_1, \ldots, \sigma(n-1) = j_{n-1}] \\ &= \frac{1}{p_1} \cdot \frac{1}{p_2} \cdots \frac{1}{p_n} \\ &= \frac{1}{X_A(\sigma)}, \end{aligned}$$

where $P[\sigma]$ represents the probability that the random path σ is chosen in the process of Algorithm 2.1. Denote all feasible paths of matrix A as $\{\sigma_1, \ldots, \sigma_N\}$ where $N = \mathrm{Per}(A)$. Hence we have

$$E[X_A] = \sum_{t=1}^{N} \frac{1}{X_A(\sigma_t)} \cdot X_A(\sigma_t) = N = \mathrm{Per}(A). \qquad \square$$

3. Main results

Let $\lceil x \rceil$ denote the smallest integer such that $\lceil x \rceil \geqslant x$, and $\lfloor x \rfloor$ denote the largest integer such that $\lfloor x \rfloor \leqslant x$. The main result of this paper is the following.

Theorem 3.1. *Let matrix $A = [a_{ij}]$ and its row sums r_i, $i = 1, 2, \ldots, n$ be given as in Theorem* 1.1. *Then*

$$\mathrm{Per}(A)^2 \leqslant \prod_{i=1}^{n} a_i (r_i - a_i + 1),$$

where $a_i = \min\left\{\lceil \frac{r_i+1}{2} \rceil, \lceil \frac{i}{2} \rceil\right\}$.

Proof. Matrix $B = [b_{ij}]$ is defined such that $b_{ij} = a_{n-i+1,j}$. Hence $\mathrm{Per}(A) = \mathrm{Per}(B)$. By the estimator in Algorithm 2.1, every feasible path $\sigma = \{j_1, \ldots, j_n\}$

of matrix A has a dual path $\sigma' = \{j_n, \ldots, j_1\}$ of matrix B. This clearly gives a one-to-one correspondence between σ and σ'.

Denote $S_i = \{j | a_{ij} = 1, \quad 1 \leqslant j \leqslant n\}$, $p_i = \left|S_i \setminus \bigcup_{t=1}^{i}\{j_t\}\right|$ and $p_i' = \left|S_i \setminus \bigcup_{t=i}^{n}\{j_t\}\right|$. Then one gets

$$X_A(\sigma) = \prod_{i=1}^{n} p_i, \quad X_B(\sigma') = \prod_{i=1}^{n} p_i'.$$

Note that $p_i + p_i' = r_i + 1$, so we have

$$X_B(\sigma') = \prod_{i=1}^{n}(r_i - p_i + 1),$$

and hence

$$X_A(\sigma)X_B(\sigma') = \prod_{i=1}^{n} p_i(r_i - p_i + 1) \leqslant \prod_{i=1}^{n} a_i(r_i - a_i + 1),$$

where $a_i = \min\left\{\lceil \frac{r_i+1}{2} \rceil, \ i, \ n-i+1\right\}$. Rearrange the rows of matrix A in the order of $\{1, n, 2, n-1, \ldots, i, n-i+1, \ldots\}$, the corresponding a_i can be rewritten as $\min\left\{\lceil \frac{r_i+1}{2} \rceil, \ \lceil \frac{i}{2} \rceil\right\}$. Denote $\mathrm{Per}(A) = N$, we have

$$\begin{aligned}
\mathrm{Per}(A)^2 = N^2 &\leqslant \frac{N}{\frac{1}{X_A(\sigma_1)} + \cdots + \frac{1}{X_A(\sigma_N)}} \cdot \frac{N}{\frac{1}{X_B(\sigma_1')} + \cdots + \frac{1}{X_B(\sigma_N')}} \\
&\leqslant \sqrt[N]{X_A(\sigma_1) \cdots X_A(\sigma_N)} \sqrt[N]{X_B(\sigma_1') \cdots X_B(\sigma_N')} \\
&\leqslant \prod_{i=1}^{n} a_i(r_i - a_i + 1). \qquad \square
\end{aligned}$$

Theorem 3.2. *Let matrix $A = [a_{ij}]$ and its row sums r_i, $i = 1, 2, \ldots, n$ be given as in Theorem* 1.1, *and a_i be as in Theorem* 3.1. *Then*

$$\prod_{i=1}^{n} a_i(r_i - a_i + 1) \leqslant \left(\prod_{i=1}^{n} \frac{r_i + 1}{2}\right)^2.$$

Proof. Assume $r_i \in \mathbb{N}$, $1 \leqslant i \leqslant n$. For any $1 \leqslant i \leqslant n$, it is easy to show that

$$a_i(r_i - a_i + 1) \leqslant \left(\frac{r_i + 1}{2}\right)^2,$$

where $a_i = \min\left\{\lceil \frac{r_i+1}{2} \rceil, \lceil \frac{i}{2} \rceil\right\}$. Hence

$$\prod_{i=1}^{n} a_i(r_i - a_i + 1) \leqslant \left(\prod_{i=1}^{n} \frac{r_i + 1}{2}\right)^2. \qquad \square$$

Theorem 3.2 shows that the upper bound given by Theorem 3.1 is always tighter than that of Theorem 1.1. Hence Theorem 3.1 gives a refinement of Minc's result.

Numerical experiments show that the bound given by Theorem 3.1 is apparently tighter than that of Theorem 1.2 for a large proportion of matrices when n is relative small, though this proportion decreases as n grows. The following example shows that the result of Theorem 3.1 is tighter than that of Theorem 1.2 for some special classes of problems.

Example 3.1. Consider matrices consisting of $k(0 < k < n)$ full rows followed by $n - k$ rows with single or two 1's. Note the fact that

(i) for $r_i = 1$ or $r_i = 2$, $$\left(\prod_{i=1}^{n} a_i(r_i - a_i + 1)\right)^{\frac{1}{2}} = r_i!^{\frac{1}{r_i}},$$

(ii) $$\prod_{i=1}^{k} \lceil \tfrac{i}{2} \rceil \left(n - \lceil \tfrac{i}{2} \rceil + 1\right) < \left\{\prod_{i=1}^{n} \lceil \tfrac{i}{2} \rceil \left(n - \lceil \tfrac{i}{2} \rceil + 1\right)\right\}^{\frac{k}{n}} = (n!)^{\frac{2k}{n}}.$$

Hence the upper bound given by Theorem 3.1 is sharper than that of Theorem 1.2.

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